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A VLASOV THEORY FOR FIBER-REINFORCED BEAMS WITH THIN-WALLED OPEN CROSS SECTIONS

NELSON R. BAULD, JR. and LIH-SHYNG TZENG
Mechanical Engineering Department, Clemson University, Clemson, SC 29631, U.S.A.

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Abstract—This paper presents a Vlasov type theory for thin-walled beams with open cross sections made from midplane symmetric, fiber-reinforced laminates. A linear theory suitable for stress and deflection determination is presented first, followed by a nonlinear theory that is suitable for bifurcation and limit point stability analyses of the global buckling modes of beams under different applied loads and boundary conditions at end cross sections. Section properties defined by Vlasov, and the procedures used to determine them, remain valid for geometrically similar composite beams under more general interpretations. Additional section properties are established that characterize the coupling between the laminate bending and twisting moments. The theory reveals material symmetry conditions that a laminate in a cross section must satisfy with respect to other laminates in the cross section (in addition to geometric symmetry) in order that an uncoupling of the prebuckling bending and twisting modes, and hence bifurcation, is possible.

Thin-walled beams with open cross sections made from isotropic materials are used extensively in the aerospace industry, both as direct load carrying members and as stiffeners in panel constructions. The present intense interest in the use of laminated, fiber-reinforced materials in aerospace structures, and an emerging interest in their use in the automobile industry, suggests that a theory capable of predicting the buckling behavior of thin-walled beams with open cross sections made from fiber-reinforced laminates is needed to provide a rational basis for their design or analysis.

The purpose of this paper is to present a theory that can be used to analyze the buckling behavior of thin-walled beams with open cross sections made from fiber-reinforced laminates. The theory establishes a foundation for the analysis of either bifurcation buckling or limit point buckling and is a direct extension of the theory developed by Vlasov[1] or, more recently, by Gjelsvik[2] for isotropic, thin-walled beams with open cross sections. Derivations presented in this paper follow those presented by Gjelsvik[2] and are developed herein to substantiate the extension of the isotropic theory of thin-walled beams with open cross sections to thin-walled beams with open cross sections made from fiber-reinforced laminates.

In the isotropic theory of thin-walled beams with open cross sections primary equilibrium configurations exist for which bending and/or axial deformations can occur without simultaneous twisting. This uncoupling of the bending, axial, and twisting modes for pure bending about a principal axis of inertia, or concentric axial compression, or eccentric axial compression results in an eigenvalue problem that determines the critical value of the load parameter for which equilibrium configurations that are infinitesimally close to the primary equilibrium configuration exist. Generally, the bending and torsional displacement modes associated with the adjacent equilibrium configurations are coupled. The presence or absence of this coupling in the buckling modes depends on the relative positions of the section's centroid and its shear center. For example, the centroid and shear center coincide for doubly symmetrical cross sections leading to uncoupled Euler buckling modes with respect to the principal axes of inertia and a torsional buckling mode. On the other hand, the centroid and shear center for nonsymmetrical sections do not coincide and the bending and twisting modes associated with adjacent equilibrium configurations are coupled.

In the theory of thin-walled beams with open cross sections made from fiber-reinforced laminates primary equilibrium configurations can also exist for which bending and axial deformation modes can occur without simultaneous twisting deformations, thus leading to bifurcation analyses identical to those for isotropic beams. However, this uncoupling

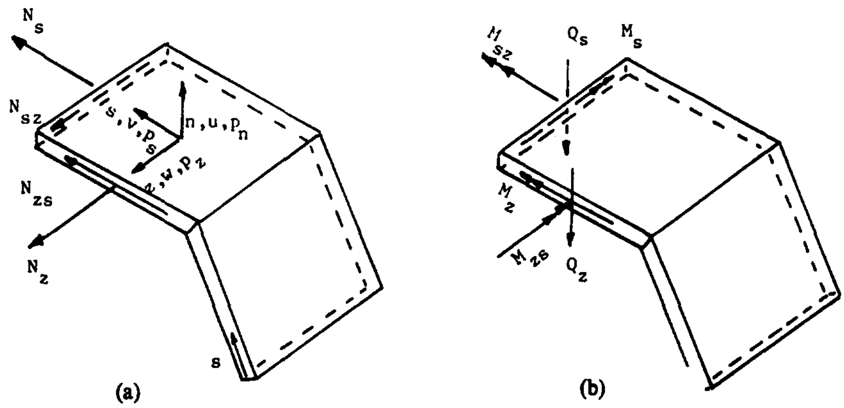

Fig. 1. Pictorial definitions of membrane force resultants, transverse shearing force resultants, bending and twisting moment resultants, middle surface displacements, and external load distributions for a plate element of a beam with a thin-walled, open cross section.

n, s, z plate element coordinate system. n is a coordinate normal to the middle surface of a plate element, s is tangent to the middle surface and is directed along the contour of the cross section, and z is parallel to the beam axis.

u, v, w middle surface displacement components in the n, s, and z directions, respectively.

p_n, p_s, p_z middle surface load distributions in the n, s, and z directions, respectively.

N_s, N_z, N_{sz}, N_{zs} membrane force resultants for a plate element.

M_s, M_z, M_{sz}, M_{zs} moment resultants for a plate element.

Q_s, Q_z transverse shearing force resultants.

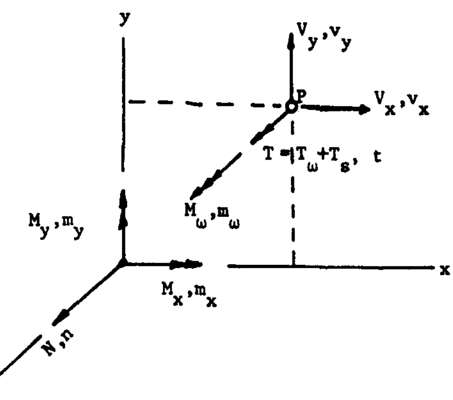

Fig. 2. Pictorial definitions of resultant beam forces, couples, and load distributions.

z axial coordinate. Same coordinate as defined in Figure 1.

x, y rectangular cartesian coordinate system in the plane of the cross section of a beam.

of bending and twisting modes for a primary equilibrium configuration appears to be limited to regular cross-ply, midplane symmetric fiber patterns. Indeed, the present theory shows that for this special fiber pattern the beam behaves in a quasi-isotropic manner and, consequently, that the buckling analyses of beams made from laminates possessing this special fiber pattern are identical to corresponding buckling analyses for isotropic beams.

For midplane symmetrical fiber patterns other than the regular cross-ply pattern, equilibrium configurations for which the bending and twisting deformation modes are uncoupled do not exist. Therefore, deformations corresponding to pure bending about a principal axis of inertia for an isotropic beam can not occur without simultaneous twisting deformations for a fiber-reinforced beam. Accordingly, a limit point analysis is required to determine the buckling behavior for such loading conditions.

Cross sections that are admitted for consideration consist of a series of connected, thin rectangular elements. Each rectangular element of a cross section can be a fiber-reinforced laminate with a fiber pattern and thickness that is different from the fiber patterns and thicknesses of other rectangular elements of the cross section. The fiber pattern of any rectangular element of a cross section is restricted to the midplane symmetric type[3]. Consequently, the coupling between the membrane force resultants and the moment resultants of a laminate vanishes.

The theoretical developments presented in this paper require three coordinate systems: an orthogonal Cartesian coordinate systems (x, y, z) for which the x and y axes lie in the plane of the cross section and the z-axis is parallel to the longitudinal generators of the beam; an orthogonal *plate* coordinate system (n, s, z) wherein the n axis is normal to the middle surface of a plate element, the s axis is tangent to the middle surface and is directed along the contour line of the cross section, and the z axis is parallel to the z axis of the x, y, z coordinate system; and a contour coordinate system s, where s is measured along the contour line of the cross section from a judiciously chosen origin. These coordinate systems are depicted in Fig. 1(a), 2 and 3, respectively.

Figures 1(a, b) depict two rectangular plate elements of a cross section. The laminate membrane forces $(N_z, N_s, N_{zs}, N_{sz})$ associated with a particular plate element are shown in Fig. 1(a), and the transverse shearing forces (Q_z and Q_s) and the moment resultants $(M_z, M_s, M_{zs}, M_{sz})$ are shown in Fig. 1(b). Plate middle surface displacements (u, v, w) and middle surface load distributions (p_n, p_s, p_z) parallel to the n, s, and z directions are shown also in Fig. 1(a). The membrane forces, transverse shearing forces, moment resultants, middle surface displacements, and middle surface load distributions are functions of the axial coordinate z and the contour coordinate s.

The essence of Vlasov's or Gjelsvik's theory of thin-walled, isotropic beams with open cross sections is the replacement of the plate stress resultants, plate displacements, and plate load distributions with equivalent systems of beam forces and couples, beam displacements, and beam load distributions that are independent of the contour coordinate

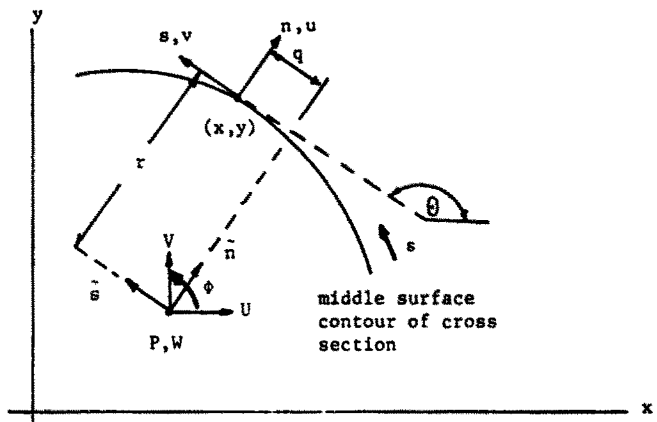

Fig. 3. Pictorial definitions of beam displacements, beam rotation, and some pertinent geometric quantities.

x_p, y_p coordinates of the pole point P.
x, y generic point on the contour of a cross section.
$\tilde{n}, \tilde{s}$ coordinate system with origin at P whose directions are parallel to those of n and s, respectively.
r, q coordinates of the point x, y on the contour measured in the $\tilde{n}$, $\tilde{s}$ coordinate systems.
θ angle the tangent to the contour makes with the x axis.
s arc length along the contour of the cross section.
N, V_x, V_y resultant beam forces: normal force and transverse shearing forces, respectively.
M_x, M_y, T resultant beam couples: bending moments and twisting torque.
T_s, T_ω, M_ω Saint Venant twisting moment, warping torque, and warping moment.
n, v_x, v_y resultant beam loads: axial force and transverse distributed forces.
t, m_x, m_y resultant beam loads: twisting moment, bending couples.
U, V, W lateral and axial beam displacements.
ϕ rotation of a cross section about the axis of the beam.

s. Gjelsvik accomplishes this equivalency between plate quantities and beam quantities through the principle of virtual work.

The equivalent beam forces (N, V_x, V_y) and beam couples (M_x, M_y, T, M_ω) are depicted in Fig. 2 where the vectors representing the normal force N and the bending moments M_x and M_y are shown acting at the origin of an arbitrary Cartesian coordinate system for which the z axis is parallel to the beam axis. Similarly, the vectors representing the beam transverse shearing forces V_x and V_y, the twisting moment T, and the warping moment M_ω are shown acting at a point P in the cross section. Point P, with coordinates x_p, y_p in the x, y coordinate system, is referred to by Vlasov as a pole. It plays a central role in the theory of beams with thin-walled open cross sections.

The twisting moment T consists of two distinct contributions: the Saint Venant torque T_s and the warping torque T_ω. These quantities are also depicted in Fig. 2 in their positive senses.

Lateral beam displacements are denoted by U and V, the axial beam displacement is denoted by W, and the rotation of the cross section by Φ. The beam displacements and the rotation are shown in Fig. 3 acting at the pole P.

Equivalent beam load distributions are shown in Fig. 2 and are denoted by n, m_x, m_y, v_x, v_y, t, and m_ω.

FUNDAMENTAL ASSUMPTIONS

Four basic assumptions provide the foundation for a *linear* engineering theory of beams with thin-walled, open cross sections made from isotropic materials. These assumptions are retained in the present theory.

(1) The contour (middle line of a plate element) of a cross section does not deform in its plane. This assumption implies that the normal strain ϵ_s in the contour direction is small compared to the normal strain ϵ_z parallel to the beam axis.

(2) The shearing strain γ_{zs} in the middle plane of a plate element is zero for each plate element.

(3) Each plate element in a cross section behaves as a thin plate. This assumption implies that the Kirchhoff hypothesis is valid for each plate element. Accordingly, line elements that are normal to the undeformed middle surface remain normal to the deformed middle surface. Moreover, these line elements are assumed to be inextensional.

(4) The normal stress σ_s in the contour direction s is small compared to the axial stress σ_z.

The foregoing four assumptions, collectively, imply that the only nonzero strains for a plate element in a cross section are the membrane strain ϵ_z and the bending and twisting strains associated with the curvature changes κ_z and κ_{zs}.

The development of an engineering theory of beams with thin-walled, open cross sections whose plate elements are made from fiber-reinforced laminates requires two additional assumptions.

(5) Each plate element of a cross section is a thin fiber-reinforced laminate that obeys the usual laminate constitutive relations.

(6) The fiber pattern associated with each plate element of a cross section is symmetric with respect to its midplane. Consequently, the coupling between laminate membrane forces and laminate moment resultants is eliminated.

Assumptions five and six establish constitutive relations for the various plate elements in a cross section.

Two further assumptions are required to extend the linear theory of fiber-reinforced, laminated beams with open, thin-walled cross sections into the nonlinear region.

(7) Second order terms, that is, product terms in the stresses and displacements, need to be considered for only the axial stress σ_z.

(8) The plate stress resultants (N_z, M_z, M_{zs} and M_{sz}) in the displaced configuration of a cross section (called convected plate forces by Gjelsvik) are expressed in terms of plate displacements by the same equations as in the linear theory.

Assumptions one through four, and seven and eight are invoked by most investigators, including Vlasov and Gjelsvik. Since these assumptions do not involve material behavior they are assumed to remain valid for beams with thin-walled, open cross sections made from fiber-reinforced laminates. Assumptions five and six characterize the material behavior associated with midplane symmetric, fiber-reinforced laminates. These two assumptions replace Hooke's law associated with the classical isotropic theory of beams of Vlasov or Gjelsvik.

Assumptions seven and eight limit the nonlinear equilibrium equations to moderate excursions from the undeformed state. Equilibrium equations based on these assumptions are known to be adequate for bifurcation analyses of isotropic beams, since, according to the adjacent equilibrium criterion for stability, bifurcation entails equilibrium configurations that are infinitesimally close to each other. Moreover equilibrium equations based on these assumptions can be used to predict limit points provided the corresponding equilibrium configuration is not excessively remote from the undeformed configuration.

LINEAR THEORY

It is convenient to develop the linear theory of beams with thin-walled open cross sections prior to developing a nonlinear theory that is suitable for calculating bifurcation loads and limit point loads.

Kinematics

The lateral beam displacements U and V are shown in Fig. 3 acting at an arbitrary pole P. Normal and tangential displacements of points on the contour line of a cross section are denoted by u and v and are also depicted in Fig. 3 at an arbitrary point x, y on the contour. The angle the tangent to the contour at point x, y makes with the x axis of the rectangular Cartesian coordinate system x, y is denoted by θ and the rotation of a cross section is signified by Φ. From geometric considerations

$$\left.\begin{aligned} u(z,s) &= U\sin\theta - V\cos\theta - q\Phi \\ v(z,s) &= U\cos\theta + V\sin\theta + r\Phi \end{aligned}\right\}, \tag{1}$$

where U, V, and Φ are functions of the axial coordinate z alone, and θ, q, and r are functions of the contour coordinate s alone.

The axial component of displacement of an arbitrary point on the contour is obtained through assumption two; that is, $\gamma_{zs} = 0$ for points lying on the contour. Since $\gamma_{zs} = \partial w/\partial s + \partial v/\partial z$, assumption two and eqn (1) lead to the expression

$$\frac{\partial w}{\partial s} = -(U'\cos\theta + V'\sin\theta + r\Phi') \tag{2}$$

which, after integration with respect to s, yields

$$w(z,s) = W - U'x - V'y - \Phi'\omega. \tag{3}$$

Differentiation with respect to the axial coordinate z is signified by primes. Later, a dot over a quantity will be used to denote differentiation with respect to the contour coordinate s. The function $W = W(z)$ is a function of integration that is a measure of the axial displacement of a cross section, and

$$\omega = \int_C r\,\mathrm{d}s. \tag{4}$$

We note that x and y are the coordinates of a point on the contour C and are, therefore, functions of the contour coordinate s. All other quantities on the r.h.s. of eqn (3), except $\omega(s)$, are functions of the axial coordinate z only. Omega as defined by eqn (4) is a section property that Vlasov calls the sectorial area.

Laminate constitutive relations

The constitutive relations for midplane symmetric laminates[3, 4] are

$$\begin{bmatrix} N_z \\ N_s \\ N_{zs} \end{bmatrix} = \begin{bmatrix} A_{11} A_{12} A_{13} \\ A_{22} A_{23} \\ \text{SYM } A_{33} \end{bmatrix} \begin{bmatrix} \epsilon_z \\ \epsilon_s \\ \gamma_{zs} \end{bmatrix} \tag{5a}$$

$$\begin{bmatrix} M_z \\ M_s \\ M_{zs} \end{bmatrix} = \begin{bmatrix} D_{11} D_{12} D_{13} \\ D_{22} D_{23} \\ \text{SYM } D_{23} \end{bmatrix} \begin{bmatrix} \kappa_z \\ \kappa_s \\ \kappa_{zs} \end{bmatrix}. \tag{5b}$$

The laminate stiffness coefficients appearing in eqn (5a) and (5b) are defined by the relations

$$A_{ij} = \sum_{k=1}^{N} \bar{Q}_{ij}^{k}(h_k - h_{k-1}) \tag{6a}$$

and

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{N} \bar{Q}_{ij}^{k}(h_k^3 - h_{k-1}^3). \tag{6b}$$

Here $\bar{Q}_{ij}$ denote lamina stiffness coefficients[3, 4] and h_k are defined in Fig. 4.

Plate stress resultant/beam displacement relations

Plate stress resultants are expressed in terms of beam displacements as follows.

The membrane strains ϵ_z, ϵ_s, and γ_{zs} and the curvature changes κ_z, κ_s, and κ_{zs} associated with the middle surface of a laminate are given by the strain-displacement relations

$$\left.\begin{aligned} \epsilon_z &= w' \\ \epsilon_s &= \dot{v} \\ \gamma_{zs} &= \dot{w} + v' \end{aligned}\right\} \tag{7a}$$

and

$$\left.\begin{aligned} \kappa_z &= u'' \\ \kappa_s &= \ddot{u} \\ \kappa_{zs} &= 2\dot{u}' \end{aligned}\right\}. \tag{7b}$$

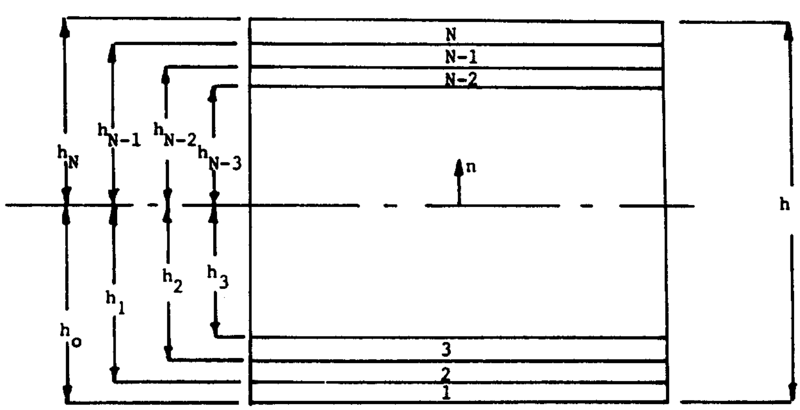

Fig. 4. Cross section of a plate element for a beam with a thin-walled, open cross section made from midplane symmetric, fiber-reinforced laminates.

Substituting u, v, and w from eqns (1) and (3) in eqns (7) yields $\epsilon_s = \gamma_{zs} = \kappa_s = 0$ and

$$\left.\begin{aligned} \epsilon_z &= W' - U''x - V''y - \Phi''\omega \\ \kappa_z &= U''\sin\theta - V''\cos\theta - \Phi''q \\ \kappa_{zs} &= -2\Phi' \end{aligned}\right\}. \tag{8}$$

Equations (5) and (8) give the laminate stress resultants N_z, M_z, M_{zs} that act at point x, y on the cross section of the contour as

$$\left.\begin{aligned} N_z &= A_{11}(W' - U''x - V''y - \Phi''\omega) \\ M_z &= D_{11}(U''\sin\theta - V''\cos\theta - \Phi''q) - 2D_{13}\Phi' \\ M_{zs} &= D_{31}(U''\sin\theta - V''\cos\theta - \Phi''q) - 2D_{33}\Phi' \end{aligned}\right\}. \tag{9}$$

The transverse shearing force resultant Q_z that acts at point x, y on the contour is obtained from the plate moment equilibrium equation[5]

$$Q_z = M_z' + \dot{M}_{zs}. \tag{10}$$

Equations (9) and (10) yield

$$Q_z = D_{11}(U'''\sin\theta - V'''\cos\theta - \Phi'''q) - (2D_{13} + D_{31})\Phi'', \tag{11}$$

where account has been taken that q is a linear function of contour coordinate s.

The membrane shearing force resultant N_{zs} is obtained directly from a free-body diagram of a finite segment of the cross sectional contour like the one shown in Fig. 5. Only the internal and external forces that enter into the equilibrium of forces parallel to the z-axis are shown. From Fig. 5

$$N_{sz}(z, s) = \int_{C^*} N_z' \, ds + \int_{C^*} p_z \, ds + \sum_{\substack{\text{over the}\\ \text{edges in } C^*}} N_{sz} + \sum_{\substack{\text{over the}\\ \text{junctions in } C^*}} T_z. \tag{12}$$

The summations that appear in eqn (12) extend over all free edges or over all junctions contained in the partial contour C^*. An edge is any extremity of a cross section that occurs naturally, while a junction is the intersection of two or more plate elements. T_z is a line load (force per unit length) applied along a junction, and p_z is a surface force (force per unit area) applied in the middle surface of a plate element.

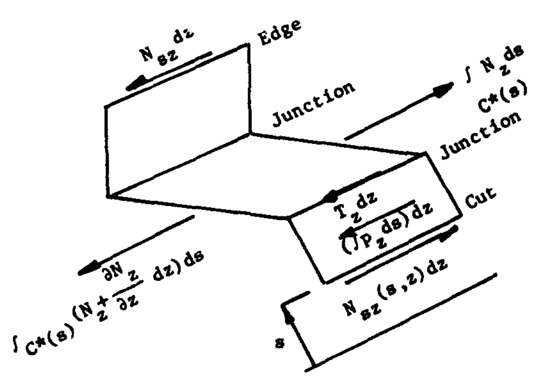

Fig. 5. Portion of beam cross section used to determine the membrane shearing force at an arbitrary point s on the contour.

The first of eqns (9) and eqn (12) lead to

$$N_{zs}(z,s) = \int_{C^*} A_{11}(W'' - U'''x - V'''y - \Phi'''\omega)\,ds + \int_{C^*} p_z\,ds + \sum_{\substack{\text{over the}\\\text{edges in } C^*}} N_{sz} + \sum_{\substack{\text{over the}\\\text{junctions in } C^*}} T_z. \tag{13}$$

We now define modified laminate stiffness coefficients through the relations

$$A_{ij} = h\sum_{k=1}^{N}\bar{Q}_{ij}^{k}\left(\frac{h_k}{h} - \frac{h_{k-1}}{h}\right) \equiv h a_{ij} \tag{14a}$$

and

$$D_{ij} = \frac{h^3}{3}\sum_{k=1}^{N}\bar{Q}_{ij}^{k}\left[\left(\frac{h_k}{h}\right)^3 - \left(\frac{h_{k-1}}{h}\right)^3\right] \equiv h^3 d_{ij}. \tag{14b}$$

We have assumed that the overall thickness h of a plate element in the cross section is constant; however, the thickness can be different for different plate elements. This implies that fiber patterns and/or laminate thicknesses may be different for the various plate elements that constitute a cross section.

With the modifications of laminate stiffness coefficients given in eqns (14), the membrane shearing force given by eqn (13) becomes

$$N_{zs}(z,s) = A^*W'' - S_y^*U''' - S_x^*V''' - S_\omega^*\Phi''' + n^* \tag{15}$$

where

$$A^* = \int_{C^*} a_{11}h\,ds$$

$$S_y^* = \int_{C^*} a_{11}xh\,ds$$

$$S_x^* = \int_{C^*} a_{11}yh\,ds$$

$$S_\omega^* = \int_{C^*} a_{11}\omega h\,ds$$

$$n^* = \int_{C^*} p_z\,dz + \sum_{\substack{\text{over the}\\\text{edges in } C^*}} N_{sz} + \sum_{\substack{\text{over the}\\\text{junctions in } C^*}} . \tag{16}$$

Here we have used a superscript star to indicate a quantity associated with a part of a contour. Observe that a_{11} is constant for a plate element; consequently, the integrals in eqn (16) can be replaced with appropriate summations. In the isotropic theory the material parameter a_{11} becomes Young's modulus of elasticity. Vlasov defines quantities $A = \int h\,ds$, $S_y = \int xh\,ds$, $S_x = \int yh\,ds$, and $S_\omega = \int \omega h\,ds$ that are geometric properties of the cross section only. For a theory applicable to beams with thin-walled, open cross sections made of fiber-reinforced, laminated materials it is necessary to redefine these properties so that they include the affect of material behavior. Physically, Vlasov's parameters are, respectively, the area, first moments of the area with respect to the y and x coordinate axes, and a property that Vlasov referred to as the first sectorial moment. When these parameters are multiplied by Young's modulus E one obtains stiffness parameters analogous to those exhibited by eqn (16).

We also note that $\omega = \int_C r \, ds$ is at most a linear function of the contour coordinate s because r is constant for each plate element.

Equations (9), (11) and (15) express the five plate stress resultants (N_z, N_{zs}, M_z, M_{zs}, and Q_z) that act along the contour of a cross section in terms of the beam displacements (W, U, V, and Φ).

Equivalent beam stress resultants

In the previous section the plate stress resultants N_z, N_{zs}, M_z, M_{zs}, and Q_z were expressed in terms of beam displacements W, U, V, and Φ for a midplane symmetric fiber-reinforced plate element. In this section we list beam stress resultants that are equivalent to the distributions of plate stress resultants acting on a cross section of a beam. Figure 6 depicts the plate stress resultants and the corresponding equivalent beam stress resultants.

The guiding principle that provides the required equivalency between plate stress resultants and beam stress resultants is the principle of virtual work[2]

$$N = \int_C N_z \, ds \tag{17a}$$

$$V_x = \int_C \left(x\frac{\partial N_z}{\partial z} - \frac{\partial M_z}{\partial z} \sin\theta \right) ds - m_y = -\frac{\partial M_y}{\partial z} - m_y \tag{17b}$$

$$V_y = \int_C \left(y\frac{\partial N_z}{\partial z} + \frac{\partial M_z}{\partial z} \cos\theta \right) ds + m_x = \frac{\partial M_x}{\partial z} + m_x \tag{17c}$$

$$M_x = \int_C (N_z y + M_z \cos\theta) \, ds \tag{17d}$$

$$M_y = -\int_C (N_z x - M_z \sin\theta) \, ds \tag{17e}$$

$$T = T_\omega + T_s = \left[\int_C \left(\omega\frac{\partial N_z}{\partial z} + q\frac{\partial M_z}{\partial z} \right) ds - m_\omega \right] - \int_C (M_{sz} + M_{zs}) \, ds. \tag{17f}$$

$$M_\omega = -\int_C (N_z \omega + M_z q) \, ds \tag{17g}$$

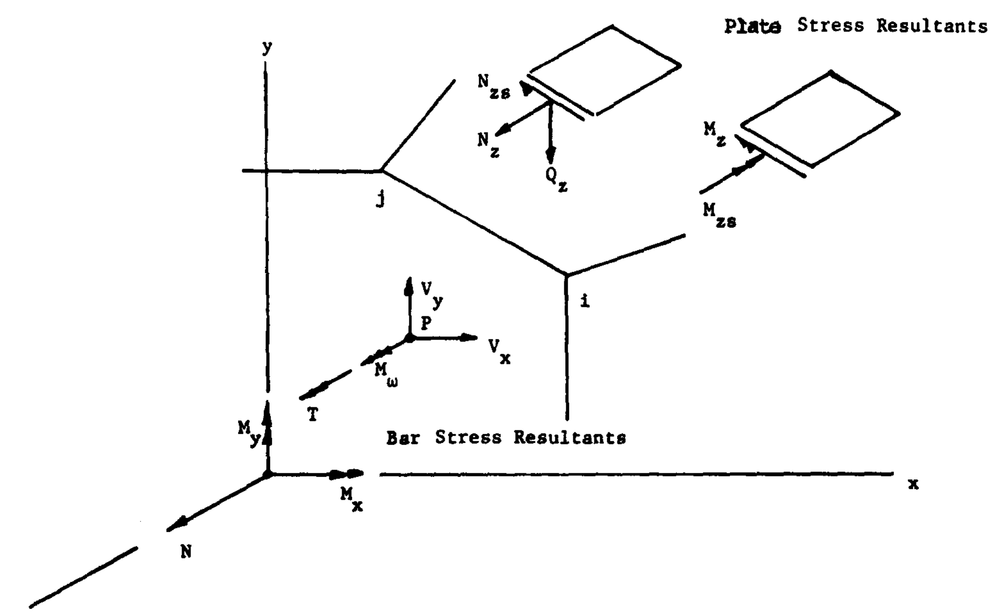

Fig. 6. Pictorial definitions of shell stress resultants and beam stress resultants.

Equivalent beam load distributions

In Fig. 7(a) p_n, p_s, and p_z denote middle surface load distributions; T_x, T_y, and T_z denote forces that are distributed along a junction; and N_{sz}, N_s, M_s, and V_s denote forces that are distributed along the free edges of a beam. These force distributions are replaced by equivalent beam force distributions n, v_x, v_y, and moment distributions m_x, m_y, t, and m_ω. The equivalence between these two sets of load distributions is accomplished through the principle of virtual work. The beam load distributions are shown acting at their appropriate points on a cross section in Fig. 7(b).

Formulas for the equivalent beam load distributions are given in Ref.[2] and are not repeated here.

Beam stress resultants-beam displacement relations

Constitutive relations for beam stress resultants N, M_x, M_y, M_ω, and T_s and beam displacements W, U, V, and Φ are obtained by substituting eqns (9) into eqns (17). Accordingly

$$\begin{bmatrix} N \\ -M_x \\ M_y \\ M_\omega \\ T_S \end{bmatrix} = \begin{bmatrix} A & -S_x & -S_y & -S_\omega & 0 \\ -S_x & I_{xx} & I_{xy} & I_{\omega x} & H_C \\ -S_y & I_{xy} & I_{yy} & I_{\omega y} & -H_S \\ -S_\omega & I_{\omega x} & I_{\omega y} & I_{\omega\omega} & H_q \\ O & H_C & -H_S & H_q & JG \end{bmatrix} \begin{bmatrix} W' \\ V'' \\ U'' \\ \Phi'' \\ \Phi' \end{bmatrix} \tag{18}$$

where

$$\left.\begin{aligned} A &= \int a_{11} h \, \mathrm{d}s \\ S_x &= \int a_{11} y h \, \mathrm{d}s \\ S_y &= \int a_{11} x h \, \mathrm{d}s \\ S_\omega &= \int a_{11} \omega h \, \mathrm{d}s \end{aligned}\right\}, \tag{19a}$$

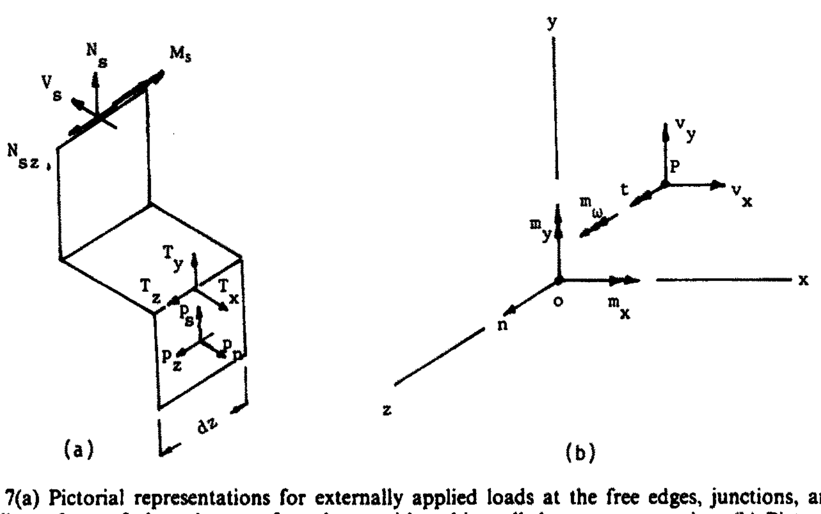

Fig. 7(a) Pictorial representations for externally applied loads at the free edges, junctions, and middle surfaces of plate elements for a beam with a thin-walled open cross section. (b) Pictorial representations for the equivalent beam load distributions.

$$\left.\begin{aligned} I_{xy} &= \int (a_{11}xy - d_{11}h^2 \sin\theta \cos\theta)h \, ds \\ I_{xx} &= \int (a_{11}y^2 + d_{11}h^2 \cos^2\theta)h \, ds \\ I_{yy} &= \int (a_{11}x^2 + d_{11}h^2 \sin^2\theta)h \, ds \\ I_{\omega x} = I_{x\omega} &= \int (a_{11}\omega y + d_{11}h^2 q \cos\theta)h \, ds \\ I_{\omega y} = I_{y\omega} &= \int (a_{11}\omega x - d_{11}h^2 q \sin\theta)h \, ds \\ I_{\omega\omega} &= \int (a_{11}\omega^2 + d_{11}h^2 q^2)h \, ds \end{aligned}\right\} \tag{19b}$$

and

$$\left.\begin{aligned} H_S &= 2\int (d_{13}h^2 \sin\theta)h \, ds \\ H_C &= 2\int (d_{13}h^2 \cos\theta)h \, ds \\ H_q &= 2\int (d_{13}h^2 q)h \, ds \\ JG &= 4\int (d_{33}h^2)h \, ds \ . \end{aligned}\right\} \tag{19c}$$

The integrals in eqns (19) cover the complete contour. It should be noted that the quantities defined in eqns (19a) and (19b) are analogous to similar quantities defined by Vlasov[1] and Gjelsvic[2] except that these new quantities reflect the behavior of the material from which the beam is made through the appearances of the modified laminate stiffness coefficients a_{11} and d_{11}. It is not difficult to show that the quantities defined in eqns (19a) and (19b) obey the same rules for coordinate transformations as the Vlasov and Gjelsvic quantities. The first three quantities defined by eqns (19c) are new quantities that do not have counterparts in the isotropic theory of thin-walled beams with open cross sections. They characterize the coupling between the bending and twisting deformation modes of a beam due to the laminated construction. The fourth quantity in eqns (19c) characterizes the torsional stiffness of the beam.

The remaining beam stress resultants V_x, V_y, and T_ω are obtained from eqns (17b), (17c), and (17f) with the aid of eqns (18). Accordingly,

$$\begin{bmatrix} V_x \\ V_y \\ T_\omega \end{bmatrix} = -\begin{bmatrix} -S_y & I_{xy} & I_{yy} & I_{\omega y} & -H_S \\ -S_x & I_{xx} & I_{xy} & I_{\omega x} & H_C \\ -S_\omega & I_{\omega x} & I_{\omega y} & I_{\omega\omega} & H_q \end{bmatrix} \begin{bmatrix} W'' \\ V''' \\ U''' \\ \Phi''' \\ \Phi'' \end{bmatrix} - \begin{bmatrix} m_y \\ -m_x \\ m_\omega \end{bmatrix}. \tag{20}$$

Equations (18) and (20) express the beam stress resultants (N, M_x, M_y, M_ω, T_s, V_x, V_y, and T_ω) in terms of beam displacements (W, U, V, Φ). They are the constitutive relations associated with beams with thin-walled, open cross sections made from fiber-reinforced laminates of the midplane symmetric type. They can be simplified considerably by appropriate choices for the origin of the Cartesian coordinate system, the origin of the contour coordinate system, and the location of the pole P. These simplifications occur in the same manner as they do for the isotropic theory thich is described in detail by Vlasov[1] or Gjelsvic[2].

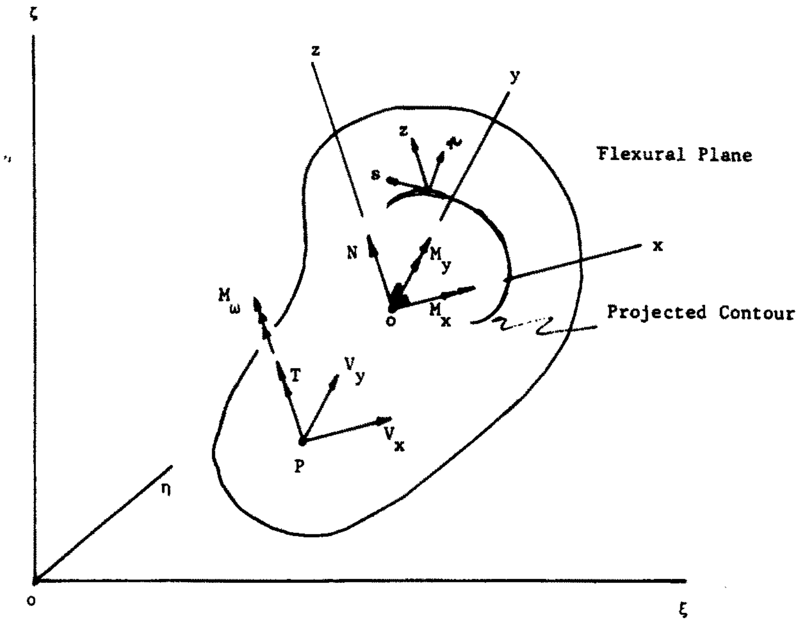

Fig. 8. Pictorial definitions of the flexural plane, projected contour line, convected beam stress resultants, and convected cartesian and contour coordinate systems.

S_x and S_y are zero when the origin of the Cartesian coordinate system coincides with the "elastic centroid" for the cross section, S_ω is zero when the origin of the contour coordinate system coincides with a principal origin, and $I_{\omega x}$ and $I_{\omega y}$ are zero when the pole for the cross section coincides with the principle pole[1, 2]. Moreover, I_{xy} is zero when the axes of the Cartesian coordinate system coincides with the axes of "principal stiffness." Consequently, the off-diagonal terms in the coefficient matrix of eqn (18), except those appearing in the last row and last column, can be reduced to zero. It is because of these nonzero off-diagonal terms that bending and twisting of beams with cross sections of the type considered can not occur independently, except for regular cross-ply fiber patterns.

Linear equilibrium equations

The linear equilibrium equations in terms of beam stress resultants are obtained from the principle of virtual work. These equations are independent of material behavior and therefore carry over from the isotropic theory unchanged. Gjelsvic[2] records these equations as

$$\left.\begin{array}{ll} N'+n=0 & M_x'-V_y+m_x=0 \\ V_x'+v_x=0 & M_y'+V_x+m_y=0 \\ V_y'+v_y=0 & M_\omega'+T-T_S+m_\omega=0 \\ T'+t=0 & \end{array}\right\}. \tag{21}$$

Elimination of the reactive beam stress resultants (V_x, V_y, T_ω) from these equations gives five equations among the active beam stress resultants (N, M_x, M_y, T_s and M_ω)

$$N'+n=0 \tag{22a}$$

$$M_x''+v_y+m_x'=0 \tag{22b}$$

$$M_y''-v_x+m_y'=0 \tag{22c}$$

$$M_\omega''-T_s'-t+m_\omega'=0. \tag{22d}$$

Equations (22) can be cast in terms of bar displacements by replacing N, M_x, M_y, T_s and M_ω with their equivalents from eqns (18). In this way one obtains the linear equilibrium equations in terms of beam displacements as

$$\begin{bmatrix} A & -S_x & -S_y & -S_\omega & 0 \\ -S_x & I_{xx} & I_{xy} & I_{\omega x} & H_C \\ -S_y & I_{xy} & I_{yy} & I_{\omega y} & -H_S \\ -S_\omega & I_{\omega x} & I_{\omega y} & I_{\omega\omega} & H_q \\ & & & & \end{bmatrix} \begin{bmatrix} W''' \\ V^{iv} \\ U^{iv} \\ \Phi^{iv} \\ \Phi''' \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & H_c & -H_S & H_q & JG \\ & & & & \end{bmatrix} \begin{bmatrix} W'' \\ V''' \\ U''' \\ \Phi''' \\ \Phi'' \end{bmatrix} = - \begin{bmatrix} n' \\ -m_x' - v_y \\ m_y' - v_x \\ m_\omega' - t \\ \end{bmatrix} . \tag{23}$$

These equations can be simplified by judicious choices for the origin and orientation of the Cartesian coordinate system, the location of the pole in this coordinate system, and the locations of the origin of the contour coordinate system. However, unlike the isotropic theory, a diagonalization of the coefficient matrices is not possible because of the H terms. Accordingly, in general, twisting will accompany bending. We note that the H terms vanish for regular cross-ply fiber patterns ($d_{13} = 0$) with the consequent complete diagonalization of the coefficient matrices.

NONLINEAR EQUILIBRIUM EQUATIONS

The stability analysis of a beam with a thin-walled, open cross section requires equilibrium equations that are based on its displaced configuration. The nonlinear equilibrium equations derived by Vlasov[1] or Gjelsvik[2] in terms of beam stress resultants remain valid for beams made from fiber-reinforced laminates. Because of the approximations invoked in their derivation, these equilibrium equations are restricted to displaced configurations that are close to the unloaded configuration.

The adjacent equilibrium stability criterion requires the existence of an equilibrium configuration that is *infinitesimally* close to, and at the same load level as, some primary equilibrium configuration whose stability is being tested. It follows that the Vlasov or Gjelsvik nonlinear equilibrium equations are adequate to calculate critical loads corresponding to bifurcated equilibrium paths.

The adequacy of these nonlinear equilibrium equations to predict limit point loads can not be taken for granted, because the limit point for a particular beam, and load, may correspond to a displaced configuration that lies beyond the range of their validity. However, neither should their applicability be overlooked.

We note that, when eqns (23) are reduced to their simplest forms ($S_x = S_y = S_\omega = I_{\omega x} = I_{\omega y} = I_{xy} = 0$), bending and twisting modes of displacement interact; even for the linear theory. This represents a significant difference between fiber-reinforced beams and classical isotropic beams. Most significantly, it changes the character of the buckling behavior from what might have been bifurcation buckling for an isotropic beam under a specific load, to limit point buckling for a fiber-reinforced beam of identical cross sectional dimensions and load.

A complete diagonalization occurs, with the consequent decoupling of bending and twisting modes of deformations, when the fiber patterns for each lamina in a cross section is of the regular cross-ply pattern. In this case $d_{13} \equiv 0$ for each lamina. It may also be possible to eliminate this coupling by adopting certain fiber patterns in combination with restricted external loading.

In any event it is essential for stability analyses of beams with thin-walled, open cross sections made from fiber-reinforced laminates that means be available to analyze both bifurcation buckling and limit point buckling.

Beam stress resultants for the nonlinear theory

Assumptions and procedures used to obtain beam stress resultants suitable for use in predicting bifurcation buckling loads and limit point loads for beams with thin-walled, open cross sections made from fiber-reinforced, midplane symmetric laminates are described in this section.

The displacement of a cross section consists of two separate modes: a displacement such that plane cross sections prior to loading remain plane after loading, and a displacement associated with warping of the cross section.

The plane corresponding to the first mode of displacement is called the flexural plane and is depicted in Fig. 8.

The warping displacement consists of the displacement of points on the contour out of the flexural plane (contour warping), and the displacement, also out of the flexural plane, of points along a normal to the contour (thickness warping).

Because of assumption one, the projection of the warped contour onto the flexural plane is the same as the original undisplaced contour. Thus, if the Cartesian xy coordinate system is assumed to be fixed in the flexural plane, the contour and its origin, pole, and centroid retain the same orientation and locations in this coordinate system as they do in the linear theory. A three dimensional coordinate system is completed by requiring the z-axis to be perpendicular to the flexural plane. The x, y, z coordinate system is therefore an orthogonal coordinate system that follows the flexural plane.

Internal *beam stress resultants* are assumed to follow the flexural plane as shown in Fig. 8. Note that V_x, V_y, and T act at the pole, and M_x, M_y, and N act at the origin of x, y, z coordinate system.

The contour coordinate system (n, s, z) is defined to be fixed to the projected contour and the flexural plane. This coordinate system has the same relationship to the projected contour as it does to the undisplaced contour in the linear theory. The *plate stress resultants* associated with the displaced beam are defined to follow the n, s, z coordinate system and are referred to as convected plate resultants.

Because the contour warps due to twisting, the material fibers of the beam do not follow the n, s, z coordinate system. The directions of material fibers in the displaced configuration that were parallel to the directions n, s, z in the undisplaced configuration are signified by n^*, s^*, z^*. Because of assumption three, n^* is perpendicular to the plane of s^* and z^*, and because of assumption two, s^* is perpendicular to z^*. Consequently, for beams with thin-walled, open cross sections n^*, s^*, z^* from an orthogonal coordinate system.

The z^* axis is perpendicular to the warped cross section. Therefore, it is rotated about the n and s axes by the small angles

$$\left.\begin{aligned} \alpha &= \Phi'(r+n) \\ \beta &= -\Phi' q \end{aligned}\right\} \tag{24}$$

as shown in Fig. 9.

By assumption seven, only second order terms associated with the axial stress σ_z^* need to be considered. Accordingly, by using eqns (24), it is found that the projections of σ_z^* on the flexural plane are

$$\left.\begin{aligned} \sigma_{zn} &= -\Phi' q \sigma_z^*, \\ \sigma_{zs} &= \Phi'(r+n)\sigma_z^* \\ \sigma_z &= \sigma_z^* \end{aligned}\right\}. \tag{25}$$

Convected plate stress resultants

The convected plate stress resultants N_{zs}, Q_z, and M_{zs} can be expressed in terms of convected plate stress resultants N_z and M_z by means of eqn (25) and the definitions for the plate stress resultants. For beams with thin-walled, open cross sections made from fiber-reinforced laminates,

$$\left.\begin{aligned} N_{zs} &= \Phi'(rN_z - M_z) \\ Q_z &= \Phi' q N_z \\ M_{zs} &= \Phi'\left(rM_z - \frac{D_{11}}{A_{11}} N_z\right) \end{aligned}\right\}. \tag{26}$$

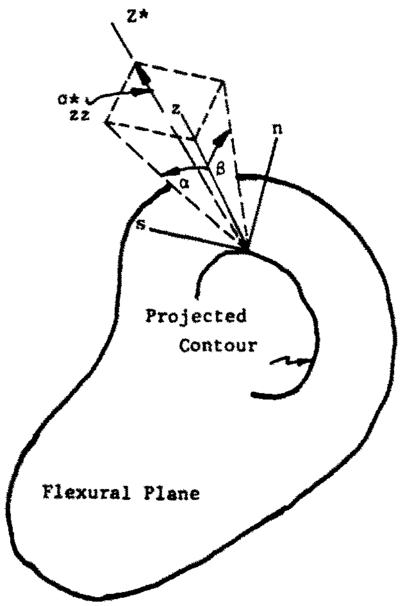

Fig. 9. Orientation of the z^* axis relative to the n, s, z coordinate system.

Equations (26) differ from the corresponding equations associated with the isotropic theory only in the last term of the third equation. This difference disappears for an isotropic material.

Convected beam stress resultants

An examination of the equivalent beam stress resultants given in eqns (17) reveals that only the beam stress resultants V_x, V_y, and T differ from those of the linear theory. These beam stress resultants consist of the linear parts given in eqns (17b), (17c), and (17f), and the nonlinear contributions that result from substituting eqns (26) into the formulas for T, V_x, and V_y given in eqns (17). This procedure leads to the formulas for the additional torque, and additional transverse shearing forces

$$\left.\begin{aligned} T &= \Phi'(AR_p^2W' - K_yI_{yy}U'' - K_xI_{xx}V'' - K_\omega I_{\omega\omega}\Phi'' + H_\Phi\Phi') \\ V_x &= \Phi'(y_pN - M_x) \\ V_y &= -\Phi'(x_pN + M_y). \end{aligned}\right\} \tag{27}$$

where

$$\begin{aligned} R_p^2 &= \frac{1}{A}\int_C a_{11}\left(r^2+q^2+\frac{D_{11}}{A_{11}}\right)h\,\mathrm{d}s \\ K_y &= \frac{1}{I_{yy}}\int_C\left[a_{11}x\left(r^2+q^2+\frac{D_{11}}{A_{11}}\right)+2rd_{11}h^2\sin\theta\right]h\,\mathrm{d}s \\ K_x &= \frac{1}{I_{xx}}\int_C\left[a_{11}y\left(r^2+q^2+\frac{D_{11}}{A_{11}}\right)-2rd_{11}h^2\cos\theta\right]h\,\mathrm{d}s \\ K_\omega &= \frac{1}{I_{\omega\omega}}\int_C\left[a_{11}\omega\left(r^2+q^2+\frac{D_{11}}{A_{11}}\right)-2rd_{11}h^2q\right]h\,\mathrm{d}s \\ H_\Phi &= \int_C 4rd_{13}h^3\,\mathrm{d}s \end{aligned} \tag{28}$$

Therefore, the total torque T, and the transverse shearing forces V_x and V_y associated with the nonlinear theory of beams with thin-walled, open cross sections made from fiber-reinforced, midplane symmetric laminates are

$$T = T_s - M'_\omega - m_\omega + \Phi'(AR_p{}^2W' - K_yI_{yy}U'' - K_xI_{xx}V'' - K_\omega I_{\omega\omega}\Phi'' + H_\phi\Phi')$$
$$V_x = -M'_y - m_y + \Phi'(y_pN - M_x)$$
$$V_y = M'_x + m_x - \Phi'(x_pN + M_y). \tag{29}$$

The remaining beam stress resultants are assumed to be given by the same equation as in the linear theory: eqns (18). Note that T, V_x, and V_y can be expressed in terms of beam displacements by means of eqn (18). Therefore, the nonlinear equilibrium equations can be expressed in terms of beam displacements U, V, W, and Φ.

Nonlinear equilibrium equations

Let ξ, η, ζ be a coordinate system that is fixed in the cross section before displacement, Fig. 8. Accordingly, the ξ, η, ζ axes are identical to the x, y, z axes of the linear theory. In the nonlinear theory the x, y, z axes follow the flexural plane as was explained previously.

The convected beam stress resultants can be expressed in the fixed ξ, η, ζ coordinate system by means of the principle of virtual work[2]. These relations are independent of material behavior and are listed here for convenience.

$$N_\zeta = N - \underline{V_xU' - V_yV'} \tag{30a}$$

$$V_\xi = V_x - V_y\Phi + NU' \tag{30b}$$

$$V_\eta = V_y + V_x\Phi + NV' \tag{30c}$$

$$M_\xi = M_x + M_y\Phi + TU' + N(V - \Phi x_p) - \underline{V_y(W - U'x_p - V'y_p)} \tag{30d}$$

$$M_\eta = M_y + M_x\Phi + TV' - N(U + \Phi y_p) + \underline{V_x(W - U'x_p - V'y_p)} \tag{30e}$$

$$T^\zeta = T + N(y_pU' - x_pV') - V_xV + V_yU - M_xU' - MyV' \tag{30f}$$

$$M_\omega{}^\zeta = M_\omega - M_xU - M_yV. \tag{30g}$$

The superscript ζ has been used with T^ζ and $M_\omega{}^\zeta$ to signify their association with the fixed ξ, η, ζ coordinate system.

For columns where N_ζ is important the shearing forces V_x and V_y are small so that the underlined terms in eqn (30a) are discarded. Moreover, the underlined terms in eqn (30d) and (30e) are also discarded on the grounds that the quantities in parentheses are of the order of magnitude of the beam displacements and V_x and V_y are typically small so that the products are negligible.

The principle of virtual work applied to the beam stress resultants associated with the fixed coordinate system (N_ζ, V_ξ, V_η, T^ζ, M_ξ, M_η, $M_\omega{}^\zeta$) leads to the equilibrium equations[2]

$$(N_\zeta)' + n^\zeta = 0 \tag{31a}$$

$$V'_\xi + v_\xi = 0 \tag{31b}$$

$$V'_\eta + v_\eta = 0 \tag{31c}$$

$$(T^\zeta)' + t^\zeta = 0. \tag{31d}$$

The remaining three equilibrium equations are replaced by the corresponding three equilibrium equations of the linear theory according to assumption eight (see eqn 21).

Using eqns (29) (with $m_x = m_y = m_\omega = 0$) and eqns (30) (with underlined terms discarded) in eqns (31) leads to equilibrium equations referred to a displaced configuration in terms of the convected beam stress resultants. Therefore,

$$N' + n = 0 \tag{32a}$$

$$M_y'' + (\Phi M_x)'' - [N(U + y_p\Phi)']' - v_x = 0 \tag{32b}$$

$$M_x'' - (\Phi M_y)'' + [N(V - x_p\Phi)']' + v_y = 0 \tag{32c}$$

$$T_s' - M_\omega'' + [N(y_pU - x_pV)']' - M_xU'' - M_yV'' + [\Phi'(AR_p^2W' - K_yI_{yy}U'' - K_xI_{xx}V'' - K_\omega I_{\omega\omega}\Phi'' + H_\Phi\Phi')]' - (v_xe_x + v_ye_y)\Phi = 0. \tag{32d}$$

Terms of the type $(M_x'' + v_x)U$ and $(M_y'' - v_y)V$ have been discarded because the quantities in parentheses are zero according to assumption 8: that is, the three equilibrium equations for the nonlinear theory that corresponds to eqns (22b)–(22d) of the linear theory are assumed to be identical to their counterparts in the linear theory. Within the context of the present theory the quantities in parentheses can not acquire magnitudes much different from zero (assuming $m_x = m_y = m_\omega = 0$) and hence their products with the beam displacements U and V constitute higher order terms.

The quantities e_x and e_y are eccentricities of the beam load distributions v_x and v_y, respectively, in the undisplaced configuration, and are measured from the pole of the cross section.

Bifurcation buckling equations

To detect bifurcation points for beams with thin-walled, open cross sections made from fiber-reinforced, midplane symmetric laminates, the beam displacements (U, V, W, ϕ) and the active beam stress resultants (N, M_x, M_y, M_ω, T_s) are expanded in asymptotic series of the form $U = U_0 + \epsilon\tilde{U}$, where ϵ is a small positive parameter.

The subscript zero is used to denote prebuckling quantities, while the tilda over a quantity signifies additional beam displacements and additional beam stress resultants that accompany the transition of the beam from the prebuckled equilibrium configuration to the buckled equilibrium configuration.

It was noted earlier in this paper that even if the prebuckled state is assumed to be linear, bending and twisting displacement modes are usually coupled for the composite thin-walled beams considered in this paper. However, there exist certain combinations of external loading and cross sectional properties that admit bifurcation buckling. For example, regular cross-ply fiber-patterns result in $d_{13} = 0$, which makes $H_S = H_C = H_q = H_\Phi = 0$ and eliminates the coupling. Consequently, buckling of beams with thin-walled, open cross sections made from midplane symmetric, regular cross-ply laminates proceeds as for isotropic beams.

Following customary procedures eqn (32) yield a set of nonhomogeneous prebuckling equilibrium equations, and a set of homogeneous buckling equations. Accordingly,

Prebuckling equilibrium equations ($T_{s0} = M_{\omega 0} = \Phi_0 = 0$)

$$N_0' + n = 0 \tag{33a}$$

$$-M_{yo}'' + \underline{(N_oU_0')'} + v_x = 0 \tag{33b}$$

$$M_{xo}'' + \underline{(N_oV_o')'} + v_y = 0 \tag{33c}$$

$$[N_o(y_pU_o - x_pV_0)']' - M_{xo}U_o'' - M_{yo}V_o'' = 0. \tag{33d}$$

Buckling equations

$$\tilde{N}' = 0 \tag{34a}$$

$$-\tilde{M}_y'' - (\tilde{\Phi}M_{xo})'' + [N_o(\tilde{U} + y_p\tilde{\Phi})']' + \underline{(\tilde{N}U_o')'} = 0 \tag{34b}$$

$$\tilde{M}_x'' - (\tilde{\Phi} M_{yo})'' + [N_o(\tilde{V} - x_p\tilde{\Phi})']' + \underline{(\tilde{N}V_o')'} = 0 \quad (34c)$$

$$\tilde{T}_s' - \tilde{M}_\omega'' + [N_o(y_p\tilde{U} - x_p\tilde{V})']' - M_{xo}\tilde{U}'' - M_{yo}\tilde{V}'' + \underline{[\tilde{N}(y_pU_o - x_pV_o)']'}$$
$$-(v_xe_x + v_ye_y)\tilde{\Phi} - \underline{\tilde{M}_xU_o''} - \underline{\tilde{M}_yV_o''} + \tilde{T}' = 0 \quad (34d)$$

with

$$\tilde{T} = \tilde{\Phi}'(AR_p^2W_o' - K_yI_{yy}U_o'' - K_xI_{xx}V_o''). \quad (35)$$

For linear prebuckling behavior the underlined terms in eqns (33b) and (33c) are discarded. Also the underlined terms in eqns (34) are discarded on the grounds that the buckling quantities ($\tilde{N}$, $\tilde{M}_x$, $\tilde{M}_y$) are infinitesimal quantities of the same order of magnitude as the prebuckling displacements so that products of prebuckling displacements and the buckling stress resultants are negligibly small.

SPECIAL CASES

In the following special cases the prebuckling behavior is assumed to be linear, in which instances, the underlined terms in eqns (33b) and (33c) are discarded. Moreover, it is assumed that the cross sectional properties are such that quantities H_S, H_C, H_q, and H_Φ are zero. These two assumptions assure that the bending and twisting displacement modes are uncoupled for the primary equilibrium path. Further, it is assumed that the Cartesian x, y coordinate system coincides with the principal axes of stiffness for the cross section, and its origin coincides with the "elastic centroid" for the cross section. Finally, the pole P is assumed to coincide with the principle pole for the cross section, and that ω is computed using a principal contour origin. Under these conditions the coefficient matrices in eqns (18) and (20) assume their simplest forms ($S_x = S_y = S_\omega = I_{xy} = I_{\omega x} = I_{\omega y} = 0$).

Buckling by concentric end forces

Consider an initially straight and untwisted column of length l under concentric end forces P. For a column loaded in this manner the prebuckling equilibrium equations and buckling equations, eqns (33) and (34), respectively, reduce to

Prebuckling equilibrium equations

$$N_o' = 0 \quad (36a)$$

$$M_{yo}'' = 0 \quad (36b)$$

$$M_{xo}'' = 0 \quad (36c)$$

$$y_pU_o'' - x_pV_o'' = 0. \quad (36d)$$

Since $N_o(0) = N_o(l) = -P$, eqn (36a) leads to $N_o(z) = -P$. Also since $M_{yo}(0) = M_{yo}(l) = 0$ and $M_{xo}(0) = M_{xo}(l) = 0$, eqns (36b) and (36c) lead to $M_{yo}(z) = M_{xo}(z) = 0$. Then from the constitutive relations, eqns (18), it is found that $U_o(z) = V_o(z) = 0$.

To summarize, for the prebuckled state, $N_0(z) = -P$, $M_{xo}(z) = M_{yo}(z) = 0$, $U_o(z) = V_o(z) = \Phi_o(z) = 0$, and $W_o = -Pz/A$.

Buckling equations

$$\tilde{M}_y'' + P(\tilde{U}'' + y_p\tilde{\Phi}'') = 0 \quad (37a)$$

$$\tilde{M}_x'' - P(\tilde{V}'' - x_p\tilde{\Phi}'') = 0 \quad (37b)$$

$$\tilde{T}_s' - \tilde{M}_\omega'' - P(y_p\tilde{U}'' - x_p\tilde{V}'') + \tilde{T}' = 0 \quad (37c)$$

with

$$\tilde{T} = -PR_p^2\tilde{\Phi}'. \tag{38}$$

Use has been made of the equation $\tilde{N}'(z) = 0$ and the observation that $\tilde{N}(o) = \tilde{N}(l) = 0$ so that $\tilde{N}(z) \equiv 0$.

The constitutive relations, eqns (18), and eqns (37) lead to the homogeneous differential equations

$$I_{yy}\tilde{U}^{iv} + P(\tilde{U}'' + y_p\tilde{\Phi}'') = 0 \tag{39a}$$

$$I_{xx}\tilde{V}^{iv} + P(\tilde{V}'' - x_p\tilde{\Phi}'') = 0 \tag{39b}$$

$$I_{\omega\omega}\tilde{\Phi}^{iv} + (PR_p^2 - JG)\tilde{\Phi}'' + P(y_p\tilde{U}'' - x_p\tilde{V}'') = 0. \tag{39c}$$

For columns with doubly symmetric cross sections $x_p = y_p = 0$ and eqns (39) simplify to

$$I_{yy}\tilde{U}^{iv} + P\tilde{U}'' = 0 \tag{40a}$$

$$I_{xx}\tilde{V}^{iv} + P\tilde{V}'' = 0 \tag{40b}$$

$$I_{\omega\omega}\tilde{\Phi}^{iv} + (PR_p^2 - JG)\tilde{\Phi}'' = 0. \tag{40c}$$

Equations (39) for nonsymmetric cross sections and eqns (40) for doubly symmetric cross sections have the same structure as corresponding equations for non-symmetric and doubly symmetric isotropic beams, respectively. The only difference resides in the definitions for I_{xx}, I_{yy}, $I_{\omega\omega}$, and R_p.

Lateral buckling by end moments

Consider a beam with a cross section that is symmetrical with respect to the y axis and that is subjected to end couples applied in the yz plane. The prebuckling equilibrium equations and the buckling equations, eqns (33) and (34), respectively, reduce to:

Prebuckling equilibrium equations

$$N_o' = 0 \tag{41a}$$

$$-M_{yo}'' + \underline{[N_oU_o']'} = 0 \tag{41b}$$

$$M_{xo}'' + \underline{[N_oV_o']'} = 0 \tag{41c}$$

$$[N_o(y_pU_o - x_pV_o)]' - \underline{M_{xo}U_o''} - \underline{M_{yo}''V_o''} = 0. \tag{41d}$$

Since $N_o(o) = N_o(l) = 0$ (only bending moments M_{xo} act at the end sections) eqn (41a) leads to $N_o(z) \equiv 0$. Consequently, eqn (41b) and (41c) yield $M_{yo}''(z) = 0$ and $M_{xo}''(z) = 0$. Moreover, since $M_{yo}(o) = M_{yo}(l) = 0$, $M_{yo}(z) \equiv 0$; and because $M_{xo}(o) = M_{xo}(l) = M$, it follows that $M_{xo}(z) = M$.

Equation (41d) now yields $U_o''(z) = 0$, which requires that $U_o(z)$ be a linear function of z. If $U_o(o) = U_o(l) = 0$ then $U_o(z) \equiv 0$. The constitutive equations, eqns (18), show that $I_{xx}V_o''(x) = -M_{xo} = -M$.

To summarize, for the prebuckled state, $N_o(z) = M_{yo}(z) = U_o(z) \equiv 0$, $M_{xo}(z) = M$, and $I_{xx}V_o''(x) = -M$.

Buckling equations

$$\tilde{N}' = 0 \tag{42a}$$

$$-\tilde{M}_y'' - (\tilde{\Phi} M)'' = 0 \tag{42b}$$

$$\tilde{M}_x'' = 0 \tag{42c}$$

$$\tilde{T}_s' - \tilde{M}_\omega' - M\tilde{U}'' + \tilde{T}' = 0 \tag{42d}$$

with

$$\tilde{T} = K_x M \tilde{\Phi}'. \tag{43}$$

Again we note that $\tilde{N}(o) = \tilde{N}(l) = 0$ so that eqn (42a) leads to $\tilde{N}(z) \equiv 0$. Then using eqns (18), eqns (42b)–(42d) can be expressed in terms of the buckling displacements $\tilde{U}$, $\tilde{V}$, and $\tilde{\Phi}(\tilde{W}(z) \equiv 0$ because $\tilde{N}(z) \equiv 0)$. Accordingly,

$$I_{yy}\tilde{U}^{iv} + M\tilde{\Phi}'' = 0 \tag{43a}$$

$$I_{xx}\tilde{V}^{iv} = 0 \tag{43b}$$

$$I_{\omega\omega}\tilde{\Phi}^{iv} - JG\tilde{\Phi}'' + M(\tilde{U}'' - K_x\tilde{\Phi}'') = 0. \tag{43c}$$

Buckling by eccentric axial end forces

Let the beam of the previous section be subjected to eccentric axial compressive forces P at its ends. Let the eccentricity of these forces lie along the axis of symmetry (y axis) and denote it by e. For this loading the prebuckling equilibrium equations are given by eqns (36) or eqns (41) with the nonlinear terms discarded. These equations lead to the prebuckling quantities $N_o(z) = -P$, $M_{yo}(z) \equiv 0$, $M_{xo} = -Pe$, $W_o(z) = -Pz/A$, $U_o(z) = 0$, and $V_o''(z) = Pe/I_{xx}$. The buckling equations become

$$I_{yy}\tilde{U}^{iv} + P\{\tilde{U}'' + (y_p - e)\tilde{\Phi}''\} = 0 \tag{44a}$$

$$I_{xx}\tilde{V}^{iv} + P\tilde{V}'' = 0 \tag{44b}$$

$$I_{\omega\omega}\tilde{\Phi}^{iv} - JG\tilde{\Phi}'' + P\{(y_p - e)\tilde{U}'' + (R_p^2 + K_x e)\tilde{\Phi}''\} = 0. \tag{44c}$$

The examples presented are of a very special nature in that they assume that beam cross sections are such that H_S, H_C, H_q, and H_Φ are identically zero. As was pointed out previously, the bending and twisting displacement modes are uncoupled for the prebuckling behavior under these conditions. This is an extremely important circumstance because of the common usage of regular cross ply laminates in the fabrication of beams with thin-walled, open cross sections.

For beams made from midplane symmetric, angle ply laminates even the linearized form of the nonlinear equilibrium equations contain a coupling between the bending and twisting displacement modes. Consequently, buckling will generally take the character of a limit point analysis. Limit point analyses of the nonlinear equilibrium equations can be obtained via Newtons' correctional procedure[6] and Potters' numerical algorithm[7].

CONCLUSIONS

This paper shows how the linear and nonlinear theories for the bending and twisting of thin-walled beams with open cross sections made from isotropic materials can be extended to include thin-walled beams with open cross sections made from midplane symmetrical, fiber-reinforced laminates.

Section properties (A, S_x, S_y, S_ω, I_{xx}, I_{yy}, I_{xy}, $I_{\omega x}$, $I_{\omega y}$, $I_{\omega\omega}$, K_x, K_y, K_ω, R_p) that arise in the isotropic theory of beam with thin-walled open cross sections are shown to be special cases of more general definitions of section properties that arise in the extended theory presented in this paper.

A composite beam behaves in a quasi-isotropic manner when the laminates from which it is made have the additional property of regular cross-ply fiber patterns ($d_{13} = 0$). Stability analyses for composite beams whose laminates are characterized by this special fiber pattern parallel stability analyses for isotropic beams with identical geometric cross sections and identical external loading. Therefore, stability analyses that are characterized by bifurcation buckling for isotropic beams will also be characterized by bifurcation buckling for composite beams with the midplane symmetrical, regular cross-ply fiber patterns. Consequently, mathematical procedures used to determine bifurcation loads for isotropic beams can be used to determine bifurcation loads for composite beams.

When the laminates from which a beam is made are characterized by midplane symmetrical, angle-ply fiber patterns the bending and twisting deformations are coupled. This coupling of the bending and twisting deformation modes precludes primary equilibrium configurations for which bending can occur without twisting. Consequently, limit point analyses may be required to determine the critical loads for some external loadings that lead to bifurcation buckling for analogous isotropic beams.

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